NONSTATIC MODEL OF UNIVERSE WITH REVERSIBLE ANNIHILATION OF MATTER

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Abstract

In a recent article, it has been shown that the system of relativistic thermodynamics previously developed and applied by the author furnishes an important extension in our ideas as to the kind of processes which can take place at a finite rate and at the same time reversibly without increase in entropy. This makes it necessary to re-examine from the new point of view of relativistic thermodynamics those processes,—in particular the annihilation of matter by transformation into radiation and the flow of radiation out into space,—which have hitherto been regarded as furnishing unmistakable evidence that the entropy of the universe is increasing at an enormous rate. In the article mentioned, the importance of such a re-examination was made evident by treating the highly over-simplified model of a nonstatic universe filled solely with black-body radiation. In the present article a treatment is given by the methods of relativistic thermodynamics to a less simplified model of the universe containing a perfect monatomic gas in equilibrium with black-body radiation. Under the assumption that equilibrium conditions are always maintained between the gas and radiation, it is shown that the conversion of matter into radiation would then take place in such a universe at a finite rate and yet entirely reversibly without increase in entropy, and that this reversible annihilation would necessarily be accompanied also at a finite rate by an expansion of the universe,—that is by the kind of behaviour which appears in the actual universe to be associated with a red-shift in the light from the extra-galactic nebulae. It is also shown that an ordinary observer, who marks out with rigid meter sticks a small region in such a universe for his study, would find the matter in this region continually being converted into radiation; would find the energy content, energy density, and temperature of the region continually dropping; and would find a continuous net flow of radiation outward through the boundary of the region into surrounding space, which he would assume to be at a lower temperature than the contents of his region not only because of the direction of the net flow, but also because he would find the frequency of radiation entering his region from the outside on the average less than that of the radiation which was escaping. From the classical point of view these findings would evidently be interpreted by the observer as evidences for a continual increase in the entropy of his universe, in spite of the fact that all the processes in the model would actually be found to be taking place entirely reversibly when analyzed from the more legitimate point of view of the relativistic thermodynamics which must be used under the circumstances. The simplified model used for these considerations is of course by no means a satisfactory representation of the actual universe, and the assumption that the gas in the model immediately adjusts itself as to temperature and concentration so as to remain in equilibrium with the radiation appears arbitrary. Nevertheless, the analogy between the reversible phenomena occurring in such a model, and phenomena in the actual universe which have hitherto been regarded as necessarily irreversible, is so close as to emphasize the necessity of using relativistic rather than classical thermodynamics in order to obtain a real insight into the problem of the entropy of the universe as a whole.

§1. Purpose of the Present Article

IN A recent article[1] it has been shown that the system of relativistic thermodynamics, which I have previously developed[2] and applied,[3] furnishes an important extension in our ideas as to the kind of processes that can occur at a finite rate and at the same time reversibly without producing any increase in entropy. As an example, it would be impossible from the point of view of classical thermodynamics to carry out the actual expansion of a thermodynamic fluid reversibly and at a finite rate, since the friction of moving parts and the deficiency between the actual pressure exerted by the fluid and that which could be exerted with an infinitely slow rate of expansion would lead to an increase in entropy. Nevertheless, in relativistic thermodynamics, it is found that the *proper* volume associated with a thermodynamic fluid could increase at a finite rate, owing to a finite rate of change in the gravitational potentials $g_{\mu\nu}$, without involving any increase in entropy.

The new possibilities thus furnished may be of considerable importance in connection with the problem of the entropy of the universe as a whole, since they make it necessary to re-examine from the point of view of relativistic thermodynamics processes which were formerly regarded as furnishing unmistakable evidence for a continuous increase in the entropy of the actual universe. To illustrate this possible importance, consideration was given in the article mentioned to the highly simplified model of an expanding universe filled solely with black-body radiation. It was shown that the increase in proper volume of such a universe, with an accompanying decrease in the proper temperature of the black-body radiation filling the universe, would occur at a finite rate and nevertheless reversibly without increase in entropy. Furthermore, it was shown that there would be a number of phenomena in such a universe which would be regarded by an ordinary observer unfamiliar with the expansion of his universe as evidence for a flow of radiation away from his immediate neighborhood into colder regions of surrounding space, and hence would be interpreted from the classical point of view as leading to an increase of entropy, in spite of the fact that there would be no increase in entropy from the point of view of the relativistic thermodynamics which must be applied under the circumstances.

It is evident that such possibilities make the use of relativistic thermodynamics imperative for a correct analysis of the entropy changes taking place in the universe. The model of an expanding universe filled solely with black-body radiation, however, is lacking from the point of view of giving a representation of the real universe since it neglects the presence of matter which is such a characteristic feature of our actual surroundings. The model was chosen for the purposes of the previous article to illustrate the possibilities of relativistic thermodynamics with a minimum of mathematical com-

[1] Tolman, Phys. Rev. **37,** 1639 (1931).

[2] Tolman, Proc. Nat. Acad. Sci. **14,** 268 (1928); ibid. **14,** 701 (1928); Phys. Rev. **35,** 875 (1930); ibid. **35,** 896 (1930).

[3] Tolman, Proc. Nat. Acad. Sci. **14,** 348 (1928); ibid. **14,** 353 (1928); ibid. **17,** 153 (1931); Phys. Rev. **35,** 904 (1930); Tolman and Ehrenfest, Phys. Rev. **36,** 1791 (1930).

plexity, and in the present article we shall turn our attention to a more complicated model, containing matter as well as radiation.

The system which we shall treat in the present article will consist of a nonstatic universe containing black-body radiation and in addition a perfect monatomic gas. For the purposes of the discussion we shall accept the possibility of the conversion of matter into radiation and shall assume that the gas in our model adjusts itself, as the universe changes in size, so as to remain always in equilibrium with the radiation which is present. Under these assumptions, it will then be shown that the annihilation of matter in such a universe, that is its transformation into radiation, would take place at a finite rate entirely reversibly without increase in entropy and would necessarily be accompanied at all reasonable temperatures by an expansion of the universe, i.e., an increase in proper volume, also at a finite rate. Attention will also be turned to a consideration of the phenomena that would be found in such a universe by an ordinary observer who makes measurements on it. It will be shown that if such an observer should mark out with rigid meter sticks a region of the universe in his immediate vicinity for study, he would find the number of atoms of matter in this region continually decreasing with the time, partly to be sure owing to a net flow outwards across the boundary but more important for our present considerations partly also owing to the annihilation of matter. He would also find the energy content, energy density and temperature of his region decreasing with the time, would find a net flow of radiation outward across the boundary, and would find that the observed frequency of the radiation entering his region from the outside was on the average less than that of the radiation that was escaping. It is evident that our ordinary unsophisticated observer would be inclined to interpret such phenomena from the classical point of view as an irreversible annihilation of the matter in his immediate neighborhood accompanied by an irreversible flow of radiation into the colder depths of surrounding space, in spite of the fact that all the processes in the system would really be taking place without increase in entropy, when analyzed from the more legitimate point of view of relativistic thermodynamics.

The analogy between the experimental findings of the observer in the hypothetical model and our own findings in the actual universe is close enough to emphasize again the necessity of using relativistic rather than classical thermodynamics in analyzing the problem of the entropy of the universe as a whole. Nevertheless, the model of a nonstatic universe filled with gas in equilibrium with radiation may be very lacking from the point of view of giving a representation of the actual universe; the assumption that the gas immediately adjusts itself so as to remain always in equilibrium with the radiation present appears arbitrary; and certain other characteristics of the model will be discussed which may not appear satisfactory.

We may now turn to the detailed analysis of the problem.

§2. The Relativity Conditions for a Reversible Adiabatic Expansion

As a consequence of the relativistic generalization of the second law of thermodynamics which I have previously given, the conditions for a reversible adiabatic change in an isolated system consisting of a thermodynamic fluid at rest in the spatial coordinates x_1, x_2, x_3, but with a state depending on the time coordinate x_4, will be satisfied if the following expression holds at each point of the fluid[4]

$$\frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\right) = 0 \tag{1}$$

where ϕ_0 is the proper density of entropy as measured by a local observer stationary with respect to the fluid, and the other quantities have their usual significance. Or since the proper volume of fluid dV_0 in any given range of spatial coordinates $dx_1dx_2dx_3$ is determined by the well-known relativistic expression

$$dV_0 = \sqrt{-g}\,\frac{dx_4}{ds}\,dx_1dx_2dx_3 \tag{2}$$

this condition for reversibility can also be written in the form

$$\frac{\partial}{\partial x_4}\left(\phi_0\sqrt{-g}\,\frac{dx_4}{ds}\,dx_1dx_2dx_3\right) = \frac{\partial}{\partial x_4}(\phi_0 dV_0) = 0. \tag{3}$$

In accordance with this result, the proper density of entropy ϕ_0 can be changing reversibly at a finite rate with the time x_4, provided the proper volume dV_0 is also changing with the time at such a rate as to preserve the equality

$$\frac{\partial\phi_0}{\partial x_4}(dV_0) + \phi_0\frac{\partial}{\partial x_4}(dV_0) = 0. \tag{4}$$

The proper volume dV_0, however, is determined by the quantities $\sqrt{-g}$ and dx_4/ds and these in turn by the gravitational potentials $g_{\mu\nu}$, so that relativistic thermodynamics provides the possibility for a combined change in entropy density and gravitational field taking place reversibly and at a finite rate. It is the dependence of proper volume on the gravitational field, neglected in the classical thermodynamics, which leads to this new possibility characteristic for relativistic thermodynamics.

As shown in the previous article, if the conditions for reversibility are not satisfied the entropy of the system as a whole $\int\phi_0 dV_0$ would increase with the time, and since there are no processes by which an isolated adiabatic system could decrease its entropy, the system could never return to its original condition. On the other hand, if the reversibility conditions are satisfied, for example by the fulfillment of Eq. (1), there would be no increase in the entropy of the system as a result of the processes that take place. Hence there would be no thermodynamic hindrance to a return of the system to its original state,—this without reference, however, to the quite separate question as to whether or not the equations of motion, for the particular system involved, exhibit a periodic solution.

[4] See reference (1) §7.

§3. Composition of an Equilibrium Mixture of Gas and Radiation

In order to apply the above criterion for reversibility to the problem to be treated in the present article, we must first investigate the conditions for thermodynamic equilibrium in a mixture of gas and radiation, assuming the possibility of their interconvertibility.

To do this let us consider a system, consisting of a mixture of perfect monatomic gas and radiation, and small enough in extent to be treated by the methods of classical thermodynamics as they would be applied by a local observer using proper coordinates. Since the thermodynamic condition of such a system will evidently be completely specified by a statement of the three independent variables, energy content E, volume V, and number of molecules of gas N, we can write for the entropy S of the system in accordance with the first and second laws of classical thermodynamics, the general differential expression

$$dS = \frac{dE}{T} + \frac{p}{T}\,dV + \left(\frac{\partial S}{\partial N}\right)_{E,V} dN \tag{5}$$

where p is the pressure and T the absolute temperature.

To use this equation for determining the equilibrium concentration of gas under the assumption that matter and radiation are interconvertible, we must first substitute into it the condition that the gas present shall be in equilibrium with the radiation. This condition will evidently be given by the equation

$$\left(\frac{\partial S}{\partial N}\right)_{E,V} = 0 \tag{6}$$

since otherwise the entropy of the system could be increased at constant energy and volume by a transformation of radiation into matter or the reverse. Hence substituting Eq. (6) into (5), we have the simple expression

$$dS = \frac{dE}{T} + \frac{p}{T}\,dV \tag{7}$$

holding true if the gas is present at its equilibrium concentration.

With the help of this equation, however, we may now determine the equilibrium concentration of gas molecules present, by substituting for S, E and p the following evident expressions for their values in terms of N, T and V. For the entropy of the mixture we may write

$$S = \frac{3}{2}Nk\log T - Nk\log\frac{N}{V} + Nk\log be^{5/2} + \frac{4}{3}aVT^3 \tag{8}$$

where k is Boltzmann's constant, a is Stefan's constant, and b is a constant of the right magnitude to assure the same starting point for the entropy of the gas and radiation. The first two terms of this expression give the well-known dependence of the entropy of a perfect monatomic gas on temperature and concentration; the third term is introduced in order to give the same starting point for the entropy of the gas and radiation, the constant of proportionality being written as $k \log be^{5/2}$ to obtain simplicity of form in the final formula; and the last term is well known as giving the entropy of black-body radiation. For the energy of the mixture we may evidently write

$$E = Nmc^2 + \frac{3}{2} NkT + aVT^4 \tag{9}$$

where m is the rest mass of each atom and c the velocity of light. The first two terms give the internal and kinetic energy of the N molecules, and the last term the energy of the radiation. Finally, for the pressure we may evidently write

$$p = \frac{N}{V} kT + \frac{1}{3} aT^4 \tag{10}$$

where the first term is the pressure of the gas and the second the pressure of the radiation. On substituting these expressions (8), (9) and (10) into Eq. (7), and eliminating a considerable number of terms which mutually cancel each other, we obtain the final result

$$\left(kT \log \frac{bT^{3/2}N}{V} - mc^2\right) dN = 0 \tag{11}$$

which can itself be true in general only if we have the relation between equilibrium concentration and temperature

$$\frac{N}{V} = bT^{3/2} e^{-mc^2/kT}. \tag{12}$$

This expression for the concentration of a perfect monatomic gas in equilibrium with black-body radiation of temperature T was first obtained by Stern,[5] who based his derivation on all three laws of thermodynamics and obtained a specific value for the constant b which depended on his method of introducing the third law of thermodynamics. It was later shown by myself[6] that it could be derived, with an undetermined value of the constant b, from the first and second laws alone, thus avoiding possible uncertainties as to the correct method of introducing the third law of thermodynamics into such a problem.

As noted above the derivation of the formula assumes a system of gas and radiation of small enough extent to be treated by the methods of classical thermodynamics as they would be applied by a local observer using proper coordinates. Hence when we use the formula in our later considerations of the universe as a whole we must remember that the concentration and temperature are those which would be found at the point of interest by a local observer. To indicate this with certainty we may rewrite Eq. (12) in the form[7]

[5] Stern, Zeits. f. Electrochem. **31,** 448 (1925); Trans. Faraday Soc. **21,** 447 (1925–26).

[6] Tolman, Proc. Nat. Acad. Sci. **12,** 670 (1926).

[7] This form of equation which gives *in the presence of a gravitational field* the proper concentration of the gas in terms of the proper temperature, has previously been specifically derived

$$N_0 = bT_0^{3/2}e^{-mc^2/kT_0} \tag{13}$$

where N_0 denotes the number of molecules in unit proper volume and T_0 the proper temperature as found by a local observer, using natural coordinates at rest in the fluid.

It should also be specially noted, for use in our later considerations, that Eq. (7) for the dependence of entropy on energy and volume when the gas is present at its equilibrium concentration means that a local observer will find the proper entropy S_0 of a small definite portion of the fluid in his neighborhood, related to its proper energy E_0, volume V_0, temperature T_0 and pressure p_0 by the relation

$$dS_0 = \frac{dE_0}{T_0} + \frac{p_0}{T_0}dV_0 \tag{14}$$

provided the gas is present at its equilibrium concentration as given in Eq. (13).

§4. The Line Element for the Nonstatic Universe

We must now briefly review certain general properties of the nonstatic universe,[8] as a preparation for our consideration of the special model of a nonstatic universe filled with a mixture of gas and radiation.

The line element for a nonstatic universe filled with a uniform distribution of matter and energy can be derived[9] by treating the contents of the universe for the purposes of large scale considerations as a perfect fluid, on the basis of two requirements,—(a) that the fluid shall at all times be uniformly distributed spatially, and (b) the stability requirement that particles which are at rest in the coordinates used shall not be subject to acceleration.

The line element so obtained can be written in a variety of forms depending on the choice of coordinates. For the purposes of the present article it will be most convenient to write it in the form[10]

$$ds^2 = -\,e^{g(t)}\left(\frac{dr^2}{1 - r^2/R^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) + dt^2 \tag{15}$$

where r, θ and ϕ are the spatial coordinates, t is the time coordinate, R is a constant, and the dependence of the line element on the time is given by the exponent $g(t)$.

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for the case of the gravitational field produced by a spherical distribution of perfect fluid (Tolman, Phys. Rev. **35,** 923, 1930), and for the case of the gravitational field in a static Einstein universe (Tolman, Proc. Nat. Acad. Sci. **17,** 153, 1931).

[8] For an account of various treatments which have been given to the nonstatic line element for the universe, see Tolman, Proc. Nat. Acad. Sci. **16,** 582 (1930).

[9] Tolman, Proc. Nat. Acad. Sci. **16,** 320 (1930). See also ibid. **16,** 409 (1930) and note that the five assumptions mentioned in §2 of that article can be included under the heading of the two requirements (a) and (b) which are stated above.

[10] Tolman, Proc. Nat. Acad. Sci. **16,** 511 (1930). Compare Eq. (5) and note that the $\bar{r}$ of that article is our present r.

§5. Mechanics of the Nonstatic Universe

In accordance with the requirement (a) used in the derivation of the line element, the proper macroscopic density ρ_{00} and the proper pressure p_0 of the fluid which we take as filling the universe will be independent of the position r, θ, ϕ but may be changing with the time t. Indeed, working out the components of the energy-momentum tensor T_μ^ν which correspond to the line element (15), and equating to those for a perfect fluid we obtain as the only nonvanishing components[11]

$$8\pi T_1^1 = 8\pi T_2^2 = 8\pi T_3^3 = -8\pi p_0 = \frac{1}{R^2}e^{-g} + \ddot{g} + \tfrac{3}{4}\dot{g}^2 - \Lambda \tag{16}$$

$$8\pi T_4^4 = 8\pi\rho_{00} = \frac{3}{R^2}e^{-g} + \tfrac{3}{4}\dot{g}^2 - \Lambda \tag{17}$$

where Λ is the cosmological constant; and these equations give the dependence of pressure and density on the exponent g and its time derivatives $\dot{g}$ and $\ddot{g}$, and thus on the time itself.

Using these expressions for the components of the energy-momentum tensor we can now easily apply the principles of relativistic mechanics in the well-known form

$$\frac{\partial \mathfrak{T}_\mu^\nu}{\partial x_\nu} - \frac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\mu} = 0 \tag{18}$$

with $\mu = 1, 2, 3$ we merely obtain identities, but substituting into this equation for the case $\mu = 4$ we easily obtain[12]

$$\frac{\partial}{\partial t}\left(\rho_{00}\frac{r^2 \sin\theta e^{3g/2}}{\sqrt{1 - r^2/R^2}}\right) + p_0\frac{\partial}{\partial t}\left(\frac{r^2 \sin\theta e^{3g/2}}{\sqrt{1 - r^2/R^2}}\right) = 0 \tag{19}$$

This important result can also evidently be directly obtained by combining Eqs. (16) and (17).

§6. Measurements of Space and Time in the Nonstatic Universe

In accordance with the requirement (b) used in deriving the line element, free particles which are at rest in the coordinate system r, θ, ϕ will not be subject to acceleration but will remain at rest, and this can be directly verified by calculating the Christoffel three-index symbols which correspond to the line element (15) and substituting into the geodesic equations which govern the motion of a particle in general relativity.

As a result of the above, unconstrained observers who are at rest with respect to the coordinate system will not be accelerated but will remain permanently at rest with respect to the coordinate system and with respect to the fluid filling the universe. In accordance with the form of the line element (15), the proper time for such observers as measured by local clocks will evi-

[11] Tolman, Proc. Nat. Acad. Sci. **16,** 409 (1930). Eqs. (2).

[12] See reference 11, Eqs. (4).

dently agree with the coordinate time t. For the proper distance dl_0, however, as measured with rigid meter sticks we shall evidently have

$$dl_0 = \frac{e^{g/2}dr}{\sqrt{1 - r^2/R^2}} \tag{20}$$

for points at the coordinate distance dr in the radial direction, and

$$dl_0 = re^{g/2}d\theta \quad \text{and} \quad dl_0 = r \sin \theta e^{g/2}d\phi \tag{21}$$

for the θ and ϕ directions. Furthermore, for the proper volume dV_0 associated with a small range of coordinates we shall evidently have

$$dV_0 = \frac{r^2 \sin \theta e^{3g/2}}{\sqrt{1 - r^2/R^2}} drd\theta d\phi. \tag{22}$$

Although particles which are at rest in the r, θ, ϕ system of coordinates will remain at rest, nevertheless it is evident from Eqs. (20) and (21) that the proper distance between such particles will in general be changing with the time, since g is a function of the time. Thus for the proper distance l_0 from a particle located at the origin to a particle permanently located at the coordinate distance r, we shall have

$$l_0 = \int_0^r \frac{e^{g/2}dr}{\sqrt{1 - r^2/R^2}} = Re^{g/2} \sin^{-1} \frac{r}{R} \cdot \tag{23}$$

Also in accordance with Eq. (22), the proper volume associated with a given coordinate range will in general be changing with the time on account of its dependence on g. For the proper volume of the universe as a whole we can write

$$V_0 = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{e^{g/2}dr}{\sqrt{1 - r^2/R^2}} drd\theta d\phi = \pi^2R^3e^{3g/2}. \tag{24}$$

In accordance with this result we can regard $Re^{g/2}$ as the radius of the universe and speak of an expanding universe if g is increasing with the time and a contracting universe if g is decreasing with the time.

This completes the review of the general properties of the nonstatic universe which will be needed in connection with the discussion of our special model.

§7. General Properties of a Nonstatic Universe Filled with a Perfect Monatomic Gas in Equilibrium with Black-Body Radiation

a. The mechanics of the model. We must now turn to a consideration of our particular model of a nonstatic universe filled with a perfect monatomic gas in equilibrium with black-body radiation. The mechanics of such a universe will be completely determined by equations which we have previously given,—namely, Eqs. (16) and (17) connecting the pressure and energy density of the mixture with the quantity $g(t)$, Eqs. (9) and (10) which give the pressure and energy density in terms of the temperature and concentration of the gas, and Eq. (13) which gives the equilibrium concentration of the gas in terms of the temperature. Referring to these equations we can write

$$8\pi p_0 = -\frac{1}{R^2}e^{-g} - \ddot{g} - \tfrac{3}{4}\dot{g}^2 + \Lambda = 8\pi(N_0kT_0 + \tfrac{1}{3}aT_0^4) \tag{25}$$

$$8\pi\rho_{00} = \frac{3}{R^2}e^{-g} + \tfrac{3}{4}\dot{g}^2 - \Lambda = 8\pi\left(N_0mc^2 + \frac{3}{2}N_0kT_0 + aT_0^4\right) \tag{26}$$

$$N_0 = bT_0^{3/2}e^{-mc^2/kT_0} \tag{27}$$

where $g(t)$ is the quantity occurring in the line element (15) which determines the dependence on the time, N_0 is the proper concentration of the gas in molecules per unit volume as measured by a local observer, and T_0 is the proper temperature of the mixture.

Examining the above, we note that we have three equations for the three variables g, N_0, and T_0 as functions of the time, and hence should expect the possibility of a solution for these quantities as functions of the time, in terms of initial conditions at some particular time, and the constants R, Λ, k, a, m and b. These equations are so complicated, however, being simultaneous, second order, and non-linear, that it will not be practicable to try to find an explicit solution for them. Instead, we shall find it possible to draw from them the conclusions which will interest us without obtaining a solution.

b. Reversibility of changes in the model. First of all it is important to show that the changes taking place with the time in such a universe would be thermodynamically reversible, not leading to any increase in entropy. As indicated in §2 of this article and shown more in detail in the previous article already mentioned, the relativistic condition for reversibility will be satisfied for a system of the kind we are considering if we have Eq. (3) holding at each point in the fluid. And with our line element and coordinates this condition for reversibility can be written in the form

$$\frac{\partial}{dt}\left(\phi_0\frac{r^2\sin\theta e^{3g/2}}{\sqrt{1-r^2/R^2}}drd\theta d\phi\right) = \frac{\partial}{\partial t}(\phi_0 dV_0) = 0 \tag{28}$$

since, however, $\phi_0 dV_0$ is the proper entropy of the fluid contained in the coordinate range $drd\theta d\phi$, the condition can now be rewritten in accordance with Eq. (14) of §3 in the form

$$\frac{1}{T_0}\frac{\partial}{\partial t}\left(\rho_{00}\frac{r^2\sin\theta e^{3g/2}}{\sqrt{1-r^2/R^2}}drd\theta d\phi\right) + \frac{p_0}{T_0}\frac{\partial}{\partial t}\left(\frac{r^2\sin\theta e^{3g/2}}{\sqrt{1-r^2/R^2}}drd\theta d\phi\right) = 0 \tag{29}$$

the change in proper entropy being dependent solely on the change in proper energy and proper volume and not being effected by change in number of molecules, owing to the postulated equilibrium between the gas and radiation.

This condition for reversibility, however, is evidently met by our system since the purely mechanical Eq. (19) necessitates the truth of the condition given by Eq. (29). It should be emphasized, however, that this reversible behaviour obtains only because of our assumption that the gas adjusts itself as to concentration and temperature so as to remain always in equilibrium with the radiation. If, for example, in an expanding universe the concentration of gas should be greater than that given by Eq. (27), the quantity $(\partial S/\partial N)$ occurring in Eq. (5) would be negative and the annihilation of matter would lead to an increase in entropy. It may also be pointed out, nevertheless, that the purely mechanical Eq. (19) would be sufficient to insure reversibility in any nonstatic universe, having the line element (15), provided we assume complete equilibrium between the matter and radiation present.

c. Changes in energy density and temperature with expansion or contraction of universe. We may next obtain expressions connecting the rates at which the proper energy density and proper temperature are changing in our model with the rate at which $g(t)$ is changing with the time.

On the one hand, differentiating Eq. (26) with respect to the time we obtain

$$8\pi\frac{d\rho_{00}}{dt} = -\frac{3}{R^2}e^{-g}\dot{g} + \frac{3}{2}\ddot{g}\dot{g} = -\frac{3}{2}\left(\frac{2}{R^2}e^{-g} - \ddot{g}\right)\dot{g} \tag{30}$$

and on the other hand by adding Eqs. (25) and (26) we obtain

$$8\pi(\rho_{00} + p_0) = \frac{2}{R^2}e^{-g} - \ddot{g} \tag{31}$$

Hence by combining (30) and (31) we can write

$$\frac{d\rho_{00}}{dt} = -\frac{3}{2}(\rho_{00} + p_0)\dot{g} \tag{32}$$

which gives us a simple expression for the rate at which the proper energy density will be decreasing in an expanding universe with $\dot{g}$ positive, or increasing in a contracting universe with $\dot{g}$ negative.

In addition, substituting for ρ_{00} and p_0 in terms of N_0 and T_0 with the help of Eq. (27) we can easily obtain

$$\frac{1}{T_0}\frac{dT_0}{dt} = -\frac{3}{2}\frac{N_0mc^2 + \frac{5}{2}N_0kT_0 + \frac{4}{3}aT_0^4}{N_0mc^2\left(\frac{mc^2}{kT_0} + 3\right) + \frac{15}{4}N_0kT_0 + 4aT_0^4}\dot{g} \tag{33}$$

and since the factors on the right hand side are physically necessarily positive, we note that the proper temperature will be dropping in an expanding universe and rising in a contracting universe.

d. Relation of annihilation of matter to expansion of universe. Furthermore, we can now use this expression for the rate at which the proper temperature is changing with the time to obtain an expression for the rate of annihilation of matter. In accordance with Eqs. (22) and (27) we can write for the number of molecules N in any given coordinate range $drd\theta d\phi$ the expression

$$N = N_0 dV_0 = bT_0^{3/2}\, e^{-mc^2/kT_0} \frac{r^2 \sin\theta e^{3g/2}}{\sqrt{1 - r^2/R^2}}\, drd\theta d\phi \tag{34}$$

and since the molecules are at rest on the average with respect to the coordinates r, θ, ϕ, changes in this quantity with the time must be due to the annihilation or synthesis of matter.

Taking a logarithmic differentiation of Eq. (34) with respect to the time, we thus obtain for the fractional rate of change in the number of molecules in the universe

$$\frac{1}{N}\frac{dN}{dt} = \left(\frac{mc^2}{kT_0} + \frac{3}{2}\right)\frac{1}{T_0}\frac{dT_0}{dt} + \frac{3}{2}\dot{g} \tag{35}$$

And on substituting Eq. (33), changing signs, and simplifying, this gives for the fractional rate of annihilation of the matter in our model

$$-\frac{1}{N}\frac{dN}{dt} = \frac{3}{2}\,\frac{N_0 mc^2 + aT_0^4\left(\dfrac{4}{3}\dfrac{mc^2}{kT_0} - 2\right)}{N_0 mc^2\left(\dfrac{mc^2}{kT_0} + 3\right) + \dfrac{15}{4}N_0 kT_0 + 4aT_0^4}\,\dot{g} \tag{36}$$

In interpreting this equation it should be noted that mc^2 will presumably be very large compared with kT_0 under those circumstances which are likely to interest us, since even if we take the mass m as small as that of the electron it would take a temperature of more than 10^9 degrees absolute to make the two quantities equal. Hence for all reasonable temperatures, we may assume

$$mc^2 \gg kT_0 \tag{37}$$

and rewrite our expression for annihilation in the approximate form

$$-\frac{1}{N}\frac{dN}{dt} = \frac{3}{2}\,\frac{N_0 kT_0 + \dfrac{4}{3}aT_0^4}{N_0 mc^2 + 4aT_0^4\left(\dfrac{kT_0}{mc^2}\right)}\,\dot{g}. \tag{38}$$

Since a positive value of $\dot{g}$ means an expanding universe, we may now state the important conclusion, true for all reasonable temperatures, that the reversible transformation of matter into radiation in this model of the universe is necessarily connected with an expansion. It is perhaps well to emphasize this conclusion since in earlier articles I have called special attention to the fact that the conversion of matter into radiation would in any case lead to a nonstatic universe in which there would be either a red or a violet shift in the light from distant objects, and the present model is of special interest in giving an example in which the annihilation of matter is necessarily connected with the kind of behaviour which appears to be associated in our actual universe with the red-shift in the light from distant objects.

e. Consideration of the possibility of a periodic solution. Before leaving the general behaviour of the model that we are treating it will be interesting to investigate the possibilities for a solution in which the properties of the universe would change periodically with the time, owing to a periodic change in the time variable $g(t)$. We can do this without obtaining an actual solution of the complicated equations which are involved, and easily show that no periodic solution would be found. In order to have such a solution the quantity $g(t)$ would have to pass periodically through its minimum and maximum values. The condition for the minimum would require

$$\dot{g} = 0 \qquad \ddot{g} \geqq 0 \tag{39}$$

and the condition for a maximum

$$\dot{g} = 0 \qquad \ddot{g} \leqq 0. \tag{40}$$

On the other hand, combining Eqs. (25) and (26), solving for $\ddot{g}$, and setting $\dot{g} = 0$, we easily obtain

$$\ddot{g} = \frac{2}{3}\Lambda - \frac{8\pi}{3}(\rho_{00} + 3p_0) \tag{41}$$

as the value of $\ddot{g}$ when g is an extremum, and since in accordance with Eqs. (25), (26), (27) and (33), ρ_{00} and p_0 are both quantities which decrease as g increases, Eq. (41) is not compatible with the conditions placed by (39) and (40) on the values of $\ddot{g}$ when g has its minimum and maximum values.

Hence no periodic solution would occur in the case of our model. It should be specially emphasized, however, that the failure of this model to exhibit any periodic solutions is not due to any thermodynamic irreversibility in the behaviour of the system, and does not exclude the possibility of some form of periodic solution in the case of the actual universe.

§8. Interpretation by an Ordinary Observer of Phenomena in the Model

Turning our attention now in particular to the case of expansion, with the radius $Re^{g/2}$ increasing with the time, we can show that the special model of a universe, filled with an equilibrium mixture of radiation and perfect gas and expanding reversibly without increase in entropy, would nevertheless exhibit important phenomena which would be interpreted by an ordinary observer from the classical point of view as evidence that the entropy of the universe was increasing with the time.

To obtain a description of these phenomena, let us consider that the observer in our idealized model of the universe is located for convenience at the origin of the r, θ, ϕ system of coordinates and is provided with a rigid scale of proper length dl_0. With the help of this scale we may suppose him to mark out a small sphere of proper radius l_0 around the origin, which gives him a small region of the universe in his immediate vicinity to serve as the subject of his studies.

For the relation between the constant proper radius of this sphere and the coordinate r of its boundary we may evidently write in accordance with Eq. (20)

$$l_0 = \int_0^r \frac{e^{g/2}dr}{\sqrt{1 - r^2/R^2}} = e^{g/2}R \sin^{-1} \frac{r}{R} \tag{42}$$

and for the case in hand of a small sphere, with r very small compared with R, this gives us the approximate relation

$$r \approx l_0 e^{g/2} \tag{43}$$

Since the proper radius of the sphere l_0 is purposely taken constant by the observer, we note that the coordinate r of its boundary is a quantity which is decreasing with the time owing to the increase of g with the time in an expanding universe.

For the proper volume of the sphere, contained within the constant proper radius l_0, we can evidently write in accordance with Eq. (22)

$$\begin{aligned} V_0 &= \int_0^r \frac{4\pi r^2 e^{3g/2}}{\sqrt{1 - r^2/R^2}} dr \\ &= 4\pi e^{3g/2} R \left| -\frac{r}{2}\sqrt{R^2 - r^2} + \frac{R^2}{2} \sin^{-1} \frac{r}{R} \right|_0^r \end{aligned} \tag{44}$$

and developing this in the form of a series in r/R and neglecting higher terms, we obtain

$$\begin{aligned} V_0 &= 4\pi e^{3g/2} R^3 \left(-\frac{1}{2}\frac{r}{R} + \frac{1}{4}\frac{r^3}{R^3} + \frac{1}{16}\frac{r^5}{R^5} + \cdots \right. \\ &\qquad \left. + \frac{1}{2}\frac{r}{R} + \frac{1}{12}\frac{r^3}{R^3} + \frac{3}{80}\frac{r^5}{R^5} + \cdots \right) \\ &\approx \frac{4}{3}\pi r^3 e^{3g/2}. \end{aligned} \tag{45}$$

And substituting the value of r given by Eq. (43), we can write as a close approximation the result, which might be expected for the proper volume of the sphere in terms of its proper radius l_0,

$$V_0 \approx \frac{4}{3}\pi l_0^3 \tag{46}$$

which is a constant independent of the time.

We may now consider the nature of the observations which would be found by our observer in studying the portion of the universe lying inside this sphere of constant measured radius and volume.

First of all it should be specially emphasized that he will observe the amount of matter inside his sphere to be continually decreasing with the time, and this he will find to be due to two causes,—partly owing to the net escape of matter through the boundary of his sphere, and partly owing to the annihilation of matter within his sphere, i.e., its transformation into radiation.

To separate these two effects, we may evidently write for the number of molecules of gas N_s inside his sphere in terms of the total number of molecules N in the universe

$$N_s = N \frac{\frac{4}{3}\pi l_0^3}{\pi^2 R^3 e^{3g/2}} \tag{47}$$

where the numerator of the fraction is the constant proper volume of the sphere as given by Eq. (46) and the denominator is the total proper volume of the universe as given by Eq. (24). And carrying out a logarithmic differentiation of this with respect to the time coordinate t, which in accordance with the form of the line element (15) is also the proper time for our observer, we obtain, after changing signs,

$$-\frac{1}{N_s}\frac{dN_s}{dt} = -\frac{1}{N}\frac{dN}{dt} + \frac{3}{2}\dot{g} \tag{48}$$

where the first term is evidently the fractional rate of annihilation of matter in the universe as a whole and hence also within the sphere, and the second term gives the fractional rate of loss by escape through the boundary. Substituting for the rate of annihilation the value given by Eq. (38), this can also be rewritten in the form

$$-\frac{1}{N_s}\frac{dN_s}{dt} = \frac{3}{2}\,\frac{N_0 k T_0 + \frac{4}{3}aT_0^4}{N_0 mc^2 + 4aT_0^4\left(\frac{kT_0}{mc^2}\right)}\dot{g} + \frac{3}{2}\dot{g} \tag{49}$$

and since g will be positive in an expanding universe, we see that our observer will find the number of molecules within his sphere decreasing both because of annihilation and because of recession through the boundary.

Attention may next be called to the fact that the observer will find the energy density inside his sphere decreasing at the rate given by the previous Eq. (32)

$$\frac{d\rho_{00}}{dt} = -\frac{3}{2}(\rho_{00} + p_0)\dot{g}. \tag{50}$$

And since the proper volume of his sphere is not changing with the time he will find its energy content decreasing at the rate

$$\frac{dE_0}{dt} = -\frac{3}{2}(\rho_{00} + p_0)\left(\frac{4}{3}\pi l_0^3\right)\dot{g}. \tag{51}$$

Furthermore, in accordance with Eq. (33) he will find the temperature of the contents of his sphere dropping at the fractional rate

$$\frac{1}{T_0}\frac{dT_0}{dt} = -\frac{3}{2}\,\frac{N_0 mc^2 + \frac{5}{2}N_0 k T_0 + \frac{4}{3}aT_0^4}{N_0 mc^2\left(\frac{mc^2}{kT_0} + 3\right) + \frac{15}{4}N_0 k T_0 + 4aT_0^4}\dot{g} \tag{52}$$

Attention must also be turned to the conclusions that will be obtained by our observer as a result of his observations on the radiation inside his sphere. Owing to the continuous drop in proper temperature given by the last Eq. (52), it is evident that he will find the amount of radiation in his sphere continually decreasing with the time. This, however, will be a net result which will contain the factors, increase in radiation arising from the annihilation of matter and decrease in radiation arising from escape through the boundary. The latter of these factors, which will be of special interest to us, is due to the fact that the constant proper volume of his sphere $4\pi l_0^3/3$ becomes a progressively smaller and smaller fraction of the total proper volume of the universe $\pi^2 R^3 e^{3g/2}$, as g increases with the expansion of the universe, so that his sphere contains a progressively decreasing fraction of the total uniformly distributed radiation in the universe.

It should be noted that this escape of radiation could be directly detected and measured, if our observer should station one of his assistants on the boundary of his sphere at the fixed distance l_0 from the origin as measured with rigid meter sticks. This assistant would not be at rest in the coordinate system r, θ, ϕ, but in accordance with Eq. (43) would have the coordinate velocity

$$\frac{dr}{dt} = -\frac{1}{2} l_0 e^{-g/2} \frac{dg}{dt} = -\frac{1}{2} r\dot{g}. \tag{53}$$

Hence, since a proper observer at rest in the coordinate system would find no net flow of radiation, it is evident that this assistant would find a net flow of radiation outward, in the case of an expanding universe with $\dot{g}$ positive.

It should also be noted that this motion of the assistant on the boundary of the sphere relative to a proper observer at rest in the fluid, would evidently introduce a Doppler effect into the observations of the assistant in such a way that he would find the average frequency of the radiation entering the sphere from the outside less than that of the radiation escaping from the sphere into the surroundings.

This completes the statement of a considerable number of phenomena which would appear to our ordinary observer as evidences for a continual degradation in the state of his universe. As we have shown, this observer would find that the matter within his sphere of observation was gradually being annihilated by transformation into radiation; would find the energy density, and energy content of his sphere decreasing with the time; would find the temperature in his neighborhood continually dropping; and would find a net flow of radiation outward through the boundary of his sphere into surrounding space which he would assume to be at a lower temperature than the material in his sphere, not only because of the direction of this flow, but also because the average frequency of radiation entering the sphere from outside would be found to be less than that of the radiation escaping from the sphere into the surroundings.

It is evident that our ordinary unsophisticated observer would be inclined to interpret these findings from the classical point of view as evidence that the entropy of the universe was continually increasing, in spite of the fact that we have shown from the more legitimate point of view of relativistic thermodynamics that all the processes in the system are taking place reversibly without increase in entropy. It should be specially noted, moreover, that the phenomena observed in our simplified hypothetical model of the universe are very similar to phenomena in the actual universe which have hitherto been interpreted as unmistakable evidence for increasing entropy.

§9. Conclusion and Critique

It is evident that the simplified model which we have treated might provide considerable insight into the problem of the entropy of the actual universe, since it has been demonstrated in accordance with the principles of relativistic thermodynamics that many processes would take place in the model without increase in entropy, which would be quite similar to processes apparently taking place in the actual universe and ordinarily interpreted as leading to increases in entropy. There are, however, a number of unsatisfactory features of the model, which must be emphasized in order that we do not overestimate the progress that has been made.

In the first place it should be specially emphasized, as already pointed out in §7b, that the reversible behaviour of the model is definitely dependent on the assumption that the matter present immediately adjusts itself as to concentration and temperature so as to remain always in equilibrium with the radiation. Thus in the case of an expanding universe we assume that the matter disappears rapidly enough to maintain the equilibrium concentration, and then show that the expansion of the universe and its accompanying annihilation of matter would be reversible. We have not, however, in any sense proved that the annihilation would necessarily be reversible in the actual universe, nor investigated how great the increases in entropy might be if there were a tendency for the annihilation to lag behind the expansion. The model is thus in some ways more arbitrary than that of a nonstatic universe filled solely with radiation as discussed in the previous article, in which the expansion is necessarily reversible.

In the second place it should be pointed out that the simplified model of the universe which was used assumes the matter in the universe to be a uniformly distributed perfect monatomic gas, and thus neglects the actual presence of various different kinds of matter in the universe and the concentration of matter into stars and stellar systems which are characteristic features of the real universe. It should also be noted that the actual concentration of gas in equilibrium with black-body radiation at any reasonable temperature would be exceedingly low, if the constant b in Eq. (12) has the magnitude which Stern obtained for it by his method of introducing the third law of thermodynamics. Indeed its value would have to be enormous to overcome the great effect of the negative exponent mc^2/kT sufficiently to give appreciable concentrations at reasonable temperatures. Nevertheless, as I have pointed out in previous publications, the correct method of applying the third law of thermodynamics to processes, involving the transformation of matter into radiation, is perhaps still to be regarded as an open question.

Finally, it should again be remarked as shown in §7e that, although the model which has been used changes its properties at a finite rate reversibly without increase in entropy and hence encounters no thermodynamic obstacle which would prevent its return to an earlier condition, nevertheless the equations of motion governing it do not actually exhibit a periodic solution. However, this of course does not mean that there may not be some type of periodic solution in the actual universe.

In spite of these difficulties, however, it seems safe to assert in conclusion that the work has in any case illustrated the necessity of using relativistic rather than classical thermodynamics in analyzing the problem of the entropy of the universe as a whole.